SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class: Circle

A B 1 2 S

2024

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 2

General Instructions

Reading time – 10 minutes

Working time – 3 hours

Write using black pen

NESA approved calculators may be used

A reference sheet is provided with this paper

Marks may **NOT** be awarded for messy or badly arranged work

Unless otherwise stated, all answers should be left in simplified exact form

For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2 – 5)

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 14)

Attempt all Questions in Section II
Allow about 2 hours and 45 minutes for this section

Examiner:
External Examiner

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet for Questions 1–10.

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1 $P(x)$ is a polynomial of degree 4 with real coefficients.
Which one of the following statements must be false?

A. $P(x) = 0$ has no real roots.

B. $P(x) = 0$ has one (repeated root) and two non-real roots.

C. $P(x) = 0$ has one real root and three non-real roots.

D. $P(x) = 0$ has four real roots.

2 The complex numbers z, iz, and $z + iz$, where $z \neq 0$, are plotted in an Argand diagram, forming the vertices of a triangle.

What is the area of this triangle?

A. $|z| + |z|^2$

B. $\dfrac{|z|^2}{2}$

C. $|z|^2$

D. $\dfrac{\sqrt{3}\,|z|^2}{2}$

3 Which of the following is an antiderivative of $\dfrac{1}{x\ln(2x)}$?

A. $\frac{1}{2}\log_e\left(\log_e 2x\right)$

B. $\log_e\left(\log_e 2x\right)$

C. $2\log_e\left(\log_e 2x\right)$

D. $\log_e\left(x\log_e 2x\right)$

4 The velocity vector of a 5 kg mass moving in the Cartesian plane is given by

$$\underset{\sim}{v}(t) = 3\sin(2t)\,\underset{\sim}{i} + 4\cos(2t)\,\underset{\sim}{j},$$

where the velocity components are measured in metres per second.

During its motion, what is the magnitude of the maximum net force, in Newtons, acting on the mass?

A. 30

B. 40

C. 50

D. 70

5 A particle travelling in a straight line has velocity v m/s at time t s.

Its acceleration is given by $\dfrac{dv}{dt} = -\,0.05\left(v^2 - 5\right)$.

Its velocity is 50 m/s initially and is reduced to 3 m/s.
Which one of the following is an expression for the time taken in seconds for this to occur?

A. $-\,0.05\displaystyle\int_{50}^{3} v^2 - 5\ dv$

B. $-\,0.05\displaystyle\int_{3}^{50} v^2 - 5\ dv$

C. $20\displaystyle\int_{50}^{3} \frac{1}{v^2 - 5}\ dv$

D. $20\displaystyle\int_{3}^{50} \frac{1}{v^2 - 5}\ dv$

6 A particle is undergoing simple harmonic motion with velocity v given by

$$v^2 = 4 - (x + 1)^2.$$

Initially the particle was 1 m to the left of O moving to the right at 2 m/s.
Which one of the following could be the acceleration a ?

A. $a = 2\sin t$

B. $a = -\,2\sin t$

C. $a = 2\cos 2t$

D. $a = -\,2\cos 2t$

7 A particle's displacement, x, after t seconds is represented by the equation

$$x = 3\cos nt,$$

where n is a real constant.
The particle passes through the origin with a speed of $\sqrt{3}$ m/s.

What is the period, in seconds, of the particle's motion?

A. $\dfrac{2\pi\sqrt{3}}{3}$

B. $\dfrac{2\sqrt{3}}{3\pi}$

C. $2\sqrt{3}\,\pi$

D. $\dfrac{2\sqrt{3}}{\pi}$

8 A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to the motion has magnitude

$$\frac{1}{40}mv^2,$$

where v m/s is the speed of the particle and $g = 9.8$ m/s^2 is the acceleration due to gravity.

What is the terminal velocity of the particle?

A. 400 m/s

B. 392 m/s

C. 20 m/s

D. 19.8 m/s

9 Given that z is a complex number satisfying the equation $\operatorname{Arg}\left(1 - \dfrac{1}{z}\right) = \dfrac{\pi}{3}$,
which of the following expressions is true?

A. $\operatorname{Arg} z - \operatorname{Arg}(z-1) = \dfrac{\pi}{3}$

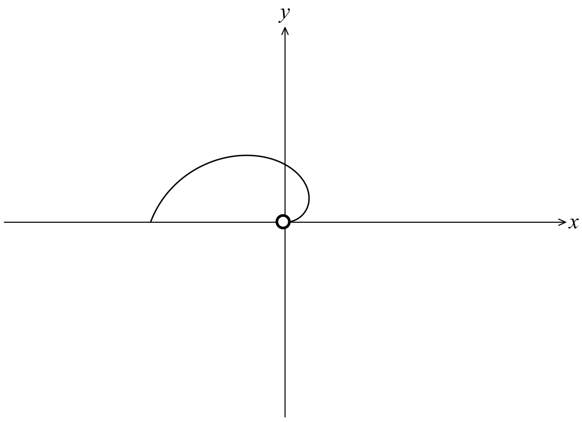
B. $\operatorname{Arg}\left(\dfrac{z-2}{z-1}\right) \geq \dfrac{\pi}{3}$

C. $|z|^2 + |z-1|^2 - |z(z-1)| - 1 = 0$

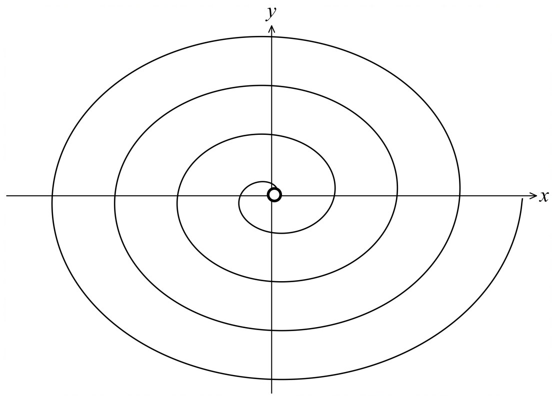
D. $|z|^2 + |z-1|^2 + |z(z-1)| - 1 = 0$

10 Which of the following best represents the complex numbers z for which $\operatorname{Arg}(z) = |z|$?

A.



B.



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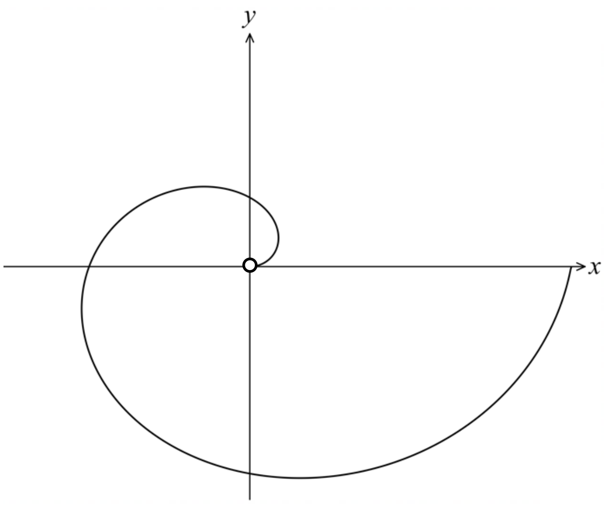
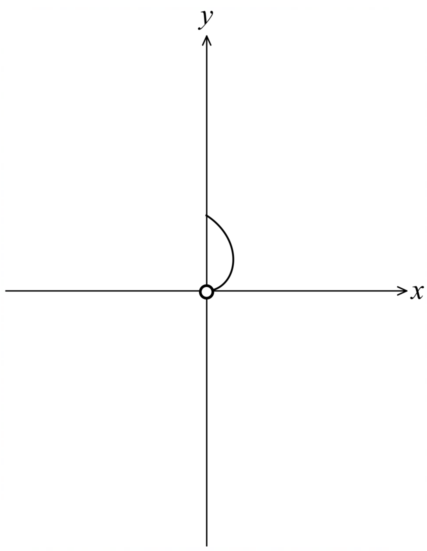


D.



Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Let $u = 2\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)$ and $v = \sqrt{3} - i$.

(i) Express uv in exponential form. **2**

(ii) If $w^2 = v$, find all possible values of w. **2**

(b) Evaluate $\displaystyle\int_1^2 x^2\sqrt{2-x}\, dx$. **3**

(c) A quartic equation

$$iz^4 + (-3-7i)z^3 + (21+17i)z^2 + (-51-15i)z + 45 = 0$$

has 4 distinct roots, z_1, z_2, z_3, and z_4 which are represented by point A, B, C, and D respectively.
It is given $z_1 = -3i$, $z_3 = 3$, and $\text{Im}\,(z_4) > 0$

(i) Find z_2 and z_4. **2**

(ii) Point E represents the complex number wz_3 such that $ABDE$ forms a parallelogram. **3**

By sketching the points A, B, and D on an Argand diagram, or otherwise, find w in the form $re^{i\theta}$ where $r > 0$, and $-\pi < \theta \leq \pi$.

Question 11 continues on page 7

(d) Sketch on the one diagram of the Argand diagram defined by **3**

$$1 \leq |z+2i| \leq 2 \text{ and } \left|\arg(z+4i)-\frac{3\pi}{8}\right| \leq \frac{\pi}{8}.$$

End of Question 11

Question 12 (14 marks) Use a SEPARATE writing booklet

(a) (i) Express $\dfrac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions. **3**

(ii) Hence prove that $\displaystyle\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx = \ln 2$. **2**

(b) Given that ω is one of the non-real roots of $z^3 = 1$, show that, $\dfrac{\omega^2}{1+\omega^2} = -\dfrac{1}{\omega^2}$. **2**

(c) A student builds a device designed to safely transport a particular fragile item that needs to be transported by dropping the item from a fixed height.

The total mass of the loaded device must be m kg and there is a parachute installed in the device which is designed to open after $\dfrac{1}{3k}$ seconds of motion.

When the parachute is opened the device experiences a resistance equal to mkv N, where k is a positive constant and v is its speed in metres per second.

It is observed that the loaded device reaches the ground and lands with a speed of $\dfrac{5g}{6k}$ m/s
Assume that without the parachute the air resistance of the device is negligible and that the acceleration due to gravity is g m/s^2.

(i) Using calculus, show that the speed of the loaded device at time $t = \dfrac{1}{3k}$ is $\dfrac{g}{3k}$ m/s. **2**

(ii) After the parachute has been opened show that $v = \dfrac{g}{k}\left(1 - \dfrac{2}{3}e^{\frac{1}{3}-kt}\right)$, for $t \geq \dfrac{1}{3k}$. **3**

(iii) Find the time taken to reach the ground, in terms of k. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) A particle moves in a straight line with acceleration $a = -5e^v$.
The particle is initially at the origin with velocity 2 m/s.

(i) Show that the particle comes to rest when $t = \frac{1}{5}\left(1 - e^{-2}\right)$. **2**

(ii) Show that the particle stops when $x = \frac{1}{5}\left(1 - 3e^{-2}\right)$. **3**

(iii) Describe the motion. **1**

(b) A sequence $u_1, u_2, u_3, \ldots$ is such that

$$u_1 = \frac{1}{4}, \text{ and } u_{n+1} = u_n + \frac{1}{n(n+1)} + 2^{-n}, \text{ for } n \in \mathbb{Z}^+.$$

(i) Prove by mathematical induction that **3**

$$u_n = \frac{9}{4} - \frac{1}{n} - 2^{-n+1}, \text{ for } n \in \mathbb{Z}^+.$$

(ii) Show that $u_n < \frac{9}{4}$ for $n \in \mathbb{Z}^+$. **1**

(c) The equation of motion for a particle is given by

$$\frac{dv}{dt} = -n^2 x,$$

where n is a positive constant, x is the displacement at time t,
and v is the velocity at time t.

(i) Show that $v^2 = n^2(A^2 - x^2)$, where A is a constant, satisfies the above equation. **1**

(ii) If the particle is initially at the origin, find the first time that the particle's speed is half its maximum speed. **4**

Question 14 (16 marks) Use a SEPARATE writing booklet

(a) A particle P is projected from the origin with initial speed V m/s at an angle 45° above the positive x-axis.

The position vector of the particle, $\mathbf{r}(t)$, where t is the time in seconds after the particle is projected, is given by

$$\mathbf{r}(t) = \begin{pmatrix} \dfrac{Vt}{\sqrt{2}} \\ \\ -\dfrac{1}{2}gt^2 + \dfrac{Vt}{\sqrt{2}} \end{pmatrix}.$$

(Do NOT prove this)

(i) Show that the equation of the trajectory of P is **2**

$$y = x - \frac{g}{V^2}x^2.$$

The point of projection (the origin) is on the floor of a barn.

The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$, and α is an acute angle.

(ii) Show that, if the particle just touches the roof then **2**

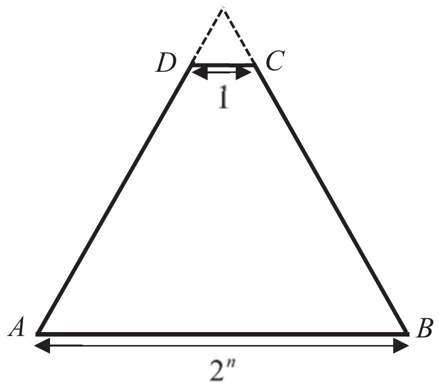
$$V(-1 + \tan\alpha) = -2\sqrt{bg}\,.$$

(iii) If this condition is satisfied, find, in terms of α, V, and g, the time after projection at which touching takes place. **2**

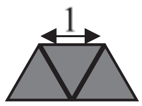
Question 14 continues on page 11

Question 14 (continued)

(b) A shape $ABCD$ is formed by taking an equilateral triangle of side length 2^n (n is a positive integer) and removing an equilateral triangle of side length 1 from one of its corners.



A trapezium tile, see diagram below, consists of three equilateral triangles of side length 1 unit.



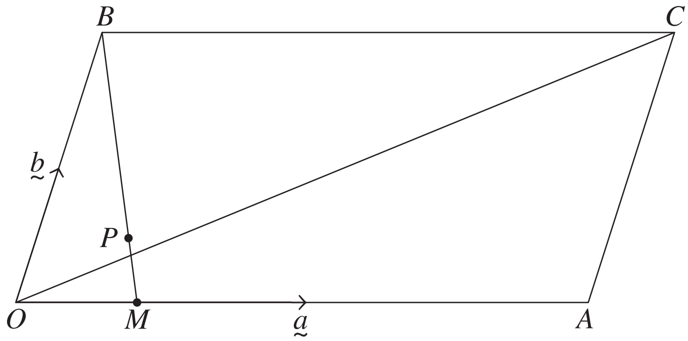
(i) Prove using induction that trapezium tiles, can fully cover the shape without any overlapping for all positive integers n. **3**

(ii) Hence explain why a regular hexagon with side length 2^n can be fully covered by these trapezium tiles, <u>and</u> determine, in terms of n, the number of tiles required. **3**

(c) Let $OACB$ be a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$

M is a point on OA such that $OM = \frac{1}{5}OA$.

P is a point on MB such that $MP = \frac{1}{6}MB$, as shown below.



(i) Show that P lies on OC. **3**

(ii) State the ratio of length $OP : PC$. **1**

End of Question 14

Question 15 (16 marks) Use a SEPARATE writing booklet

(a) A hose situated at the origin sprays water with initial speed u m/s and at angle of θ to the positive x-axis.

The hose sprays water to a height of 80 m.

We will consider the motion of one water particle in the spray.

(i) Assuming no air resistance, the position vector of the particle, $\underset{\sim}{r}(t)$, where t is the time in seconds after the water starts to spray, is given by **2**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ \\ -5t^2 + ut\sin\theta \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

Show that $u\sin\theta = 40$.

Taking into account air resistance, the acceleration, $\underset{\sim}{a}$, of a water drop is given by

$$\underset{\sim}{a}(t) = \begin{pmatrix} -0.2\dot{x} \\ \\ -10 - 0.2\dot{y} \end{pmatrix},$$

where x and y refers to the horizontal and vertical displacement respectively of a water drop at time t seconds.

(ii) Show that $y = -5(50 + u\sin\theta)e^{-0.2t} - 50t + 5(50 + u\sin\theta)$. **4**

(iii) Hence show that $y = 5\left[-\dot{y} + u\sin\theta + 50\ln\left(\dfrac{10 + 0.2\dot{y}}{10 + 0.2u\sin\theta}\right)\right]$ **3**

(iv) How high does the water from the hose reach now? Leave your answer to the nearest whole number. **1**

Question 15 continues on page 13

(b) Suppose f and g are real-valued continuous functions defined on $[0, a]$, where $a > 0$, satisfy the conditions:

$$f(x) = f(a - x) \text{ and}$$

$$g(x) + g(a - x) = m, \text{ where } m \in \mathbb{R}.$$

(i) Show that $\displaystyle\int_0^a f(x)\,g(x)\;dx = \frac{m}{2}\int_0^a f(x)\;dx$. **3**

(ii) Hence evaluate $\displaystyle\int_0^{\pi} x \sin x \cos^4 x\,dx$. **3**

End of Question 15

Question 16 is over the page

Question 16 (14 marks) Use a SEPARATE writing booklet

(a) Given $I_n = \int_0^1 (1-x)^n e^x \, dx$, where n is a non-negative integer.

(i) Show that $I_n = -1 + n\, I_{n-1}$ for $n \geq 1$. **2**

(ii) Evaluate $\int_0^1 (1-x)^3 e^x \, dx$. **2**

(b) (i) Prove that $\int_0^a f(a-x)\, dx = \int_0^a f(x)\, dx$, for constant a. **1**

(ii) Hence or otherwise, evaluate $\int_0^{2\pi} x \left|\cos x\right| dx$. **3**

(c) Let n be an integer, such that $n \neq 1$.

(i) Prove that $\sin \dfrac{\pi}{2n} \displaystyle\sum_{k=1}^{n-1} \sin \dfrac{k\pi}{n} = \cos \dfrac{\pi}{2n}$. **2**

(ii) Let $\alpha = \cos \dfrac{2\pi}{n} + i \sin \dfrac{2\pi}{n}$. **4**

Prove that $\displaystyle\sum_{k=1}^{n-1} \left|\alpha^k - 1\right| = 2 \cot \dfrac{\pi}{2n}$.

End of paper

SYDNEY
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YEAR 12

HSC TASK 4

Mathematics Extension 2

Sample Solutions

NOTE: This process of checking your mark is about reading the solutions and the comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
C	B	B	B	D	B	C	D	C	A

Section I   Multiple Choice Solutions

1 $P(x)$ is a polynomial of degree 4 with real coefficients.
Which one of the following statements must be false?

A	5
B	5
C	109
D	0

A. $P(x) = 0$ has no real roots.

B. $P(x) = 0$ has one (repeated root) and two non-real roots.

C. $P(x) = 0$ has one real root and three non-real roots.

D. $P(x) = 0$ has four real roots.

Non-real roots must occur in conjugate pairs.

2 The complex numbers z, iz, and $z + iz$, where $z \neq 0$, are plotted in an Argand diagram, forming the vertices of a triangle.

What is the area of this triangle?

A	0
B	110
C	3
D	6

A. $|z| + |z|^2$

B. $\dfrac{|z|^2}{2}$

C. $|z|^2$

D. $\dfrac{\sqrt{3}\,|z|^2}{2}$

z and iz are perpendicular and have lengths $|z|$.

3 Which of the following is an antiderivative of $\dfrac{1}{x\ln(2x)}$?

A	17
B	82
C	18
D	2

A. $\frac{1}{2}\log_e\left(\log_e 2x\right)$

B. $\log_e\left(\log_e 2x\right)$

C. $2\log_e\left(\log_e 2x\right)$

D. $\log_e\left(x\log_e 2x\right)$

$$\frac{d}{dx}(\ln(2x)) = \frac{1}{x}$$

4 The velocity vector of a 5 kg mass moving in the Cartesian plane is given by

$$\underset{\sim}{v}(t) = 3\sin(2t)\,\underset{\sim}{i} + 4\cos(2t)\,\underset{\sim}{j},$$

where the velocity components are measured in metres per second.

During its motion, what is the magnitude of the maximum net force, in Newtons, acting on the mass?

A. 30

B. 40

C. 50

D. 70

A	19
B	8
C	86
D	6

$\underset{\sim}{a} = 6\cos 2t\ \underset{\sim}{i} - 8\sin 2t\ \underset{\sim}{j}$

$$\left|\underset{\sim}{a}\right| = \sqrt{36\cos^2 2t + 64\sin^2 2t}$$

$$= \sqrt{36 + 28\sin^2 2t}$$

$\therefore\ 6 \le \left|\underset{\sim}{a}\right| \le 8$

$\therefore\ 30 \le \left|\underset{\sim}{F}\right| \le 40$

The question should have had said "maximum net force" so option A was also marked right.

5 A particle travelling in a straight line has velocity v m/s at time t s.

Its acceleration is given by $\dfrac{dv}{dt} = -0.05\left(v^2 - 5\right)$.

Its velocity is 50 m/s initially and is reduced to 3 m/s.
Which one of the following is an expression for the time taken in seconds for this to occur?

A. $-0.05\displaystyle\int_{50}^{3} v^2 - 5\ dv$

B. $-0.05\displaystyle\int_{3}^{50} v^2 - 5\ dv$

C. $20\displaystyle\int_{50}^{3} \frac{1}{v^2 - 5}\ dv$

D. $20\displaystyle\int_{3}^{50} \frac{1}{v^2 - 5}\ dv$

A	3
B	0
C	21
D	95

$$\frac{dv}{dt} = -0.05\left(v^2 - 5\right) \Rightarrow -20\,\frac{dv}{v^2 - 5} = dt$$

$$\therefore\ -20\int_{50}^{3}\frac{dv}{v^2 - 5} = \int_{0}^{T} dt \Rightarrow T = 20\int_{3}^{50}\frac{dv}{v^2 - 5}$$

6 A particle is undergoing simple harmonic motion with velocity v given by

$$v^2 = 4 - (x + 1)^2.$$

Initially the particle was 1 m to the left of O moving to the right at 2 m/s.
Which one of the following could be the acceleration a ?

A. $a = 2 \sin t$

B. $a = -2 \sin t$

C. $a = 2 \cos 2t$

D. $a = -2 \cos 2t$

A	27
B	49
C	22
D	20

$v^2 = 4 - (x + 1)^2 \Rightarrow a = -(x + 1)$

$\therefore n = 1$, so only options A and B

$v^2 = n^2(A^2 - (x + 1)^2) \Rightarrow$ Amplitude $= 2$

"Initially the particle was 1 m to the left of O moving to the right at 2 m/s." $\Rightarrow x = -1 + 2 \sin t$

7 A particle's displacement, x, after t seconds is represented by the equation

$$x = 3 \cos nt,$$

where n is a real constant.
The particle passes through the origin with a speed of $\sqrt{3}$ m/s.

What is the period, in seconds, of the particle's motion?

A. $\dfrac{2\pi\sqrt{3}}{3}$

B. $\dfrac{2\sqrt{3}}{3\pi}$

C. $2\sqrt{3}\,\pi$

D. $\dfrac{2\sqrt{3}}{\pi}$

A	27
B	9
C	82
D	1

$A = 3$

$v^2 = n^2(A^2 - x^2) \Rightarrow 3 = n^2(9 - 0)$

$\therefore n^2 = \dfrac{1}{3} \Rightarrow n = \sqrt{3}$

$\therefore T = \dfrac{2\pi}{n} = 2\sqrt{3}\,\pi$

8 A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to the motion has magnitude

$$\frac{1}{40}mv^2,$$

where v m/s is the speed of the particle and $g = 9.8$ m/s^2 is the acceleration due to gravity.

What is the terminal velocity of the particle?

A. 400 m/s

B. 392 m/s

C. 20 m/s

D. 19.8 m/s

$$m\ddot{y} = mg - \frac{1}{40}mv^2 \Rightarrow \ddot{y} = g - \frac{1}{40}v^2$$

Terminal velocity is when $\ddot{y} = 0 \Rightarrow v = \sqrt{40 \times 9.8}$

A	0
B	7
C	5
D	107

9 Given that z is a complex number satisfying the equation $\operatorname{Arg}\left(1 - \frac{1}{z}\right) = \frac{\pi}{3}$, which of the following expressions is true?

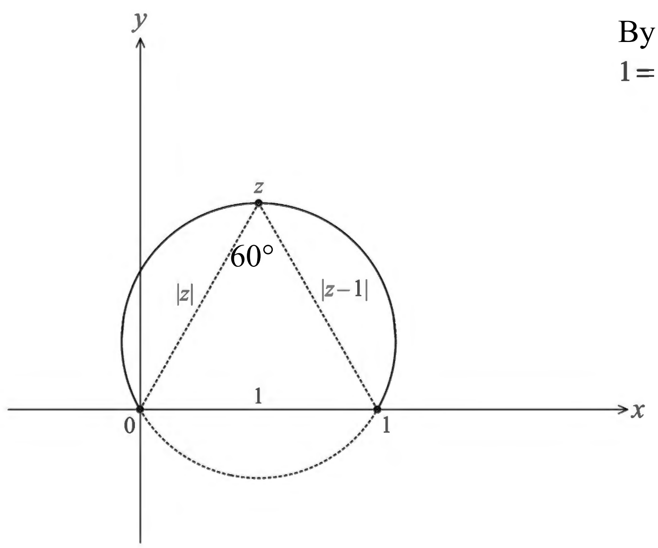
A. $\operatorname{Arg} z - \operatorname{Arg}(z-1) = \frac{\pi}{3}$

B. $\operatorname{Arg}\left(\frac{z-2}{z-1}\right) \geq \frac{\pi}{3}$

C. $|z|^2 + |z-1|^2 - |z(z-1)| - 1 = 0$

D. $|z|^2 + |z-1|^2 + |z(z-1)| - 1 = 0$

$$\operatorname{Arg}\left(1 - \frac{1}{z}\right) = \frac{\pi}{3} \Rightarrow \operatorname{Arg}\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$$

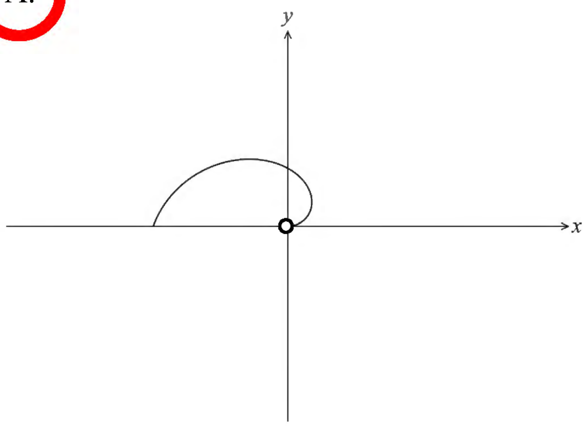


By the cosine rule:
$1 = |z|^2 + |z-1|^2 - 2 \times |z| \times |z-1| \cos 60°$

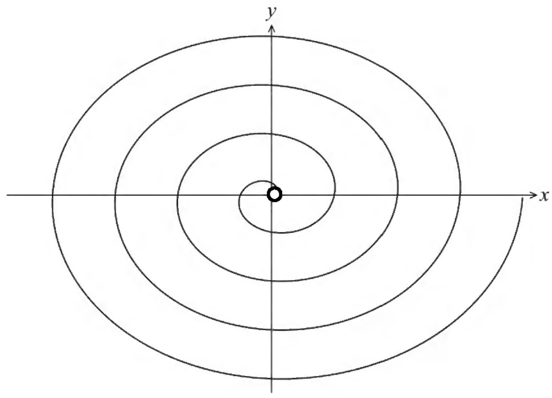
A	11
B	22
C	69
D	17

10 Which of the following best represents the complex numbers z for which $\text{Arg}(z) = |z|$?

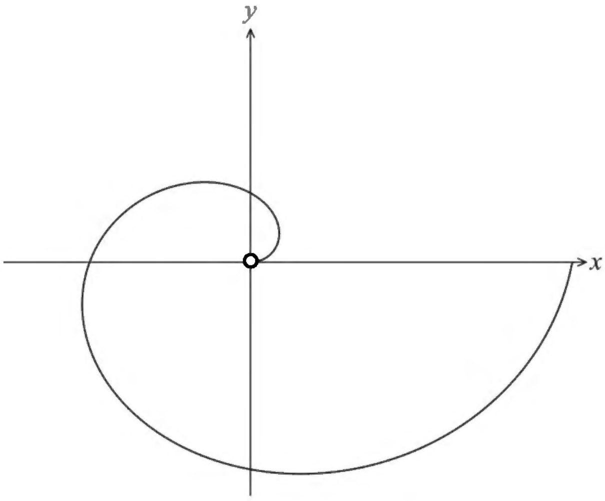
A.



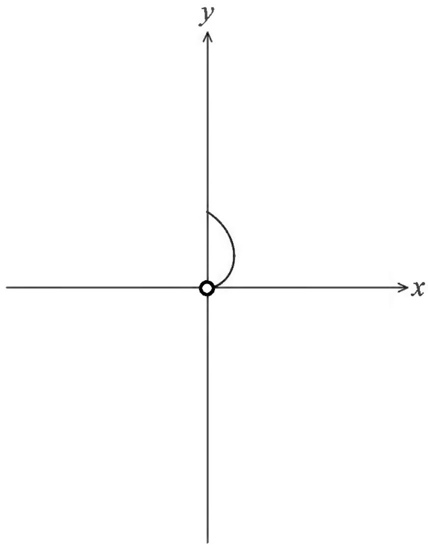
B.



C.



D.



$-\pi < \text{Arg } z \leq \pi$

$\therefore\ 0 < \text{Arg } z = |z| \leq \pi$

A	54
B	15
C	37
D	13

a) COMMENT:
Many students spent too much time on part (ii)
They failed to use the exponential form found in part (i) to reduce the required working.

(a) Let $u = 2\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)$ and $v = \sqrt{3} - i$.

(i) Express uv in exponential form. **2**

$$u = 2\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)$$
$$= 2e^{-\frac{\pi}{10}i}$$

$$v = \sqrt{3} - i$$
$$v = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$
$$v = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$= 2e^{-\frac{\pi}{6}i}$$

$$uv = \left(2e^{-\frac{\pi}{10}i}\right)\left(2e^{-\frac{\pi}{6}i}\right)$$
$$= 4e^{-\frac{4\pi}{15}i}$$

Question 11 (continued)

(a) (ii) If $w^2 = v$, find all possible values of w. **2**

$$w^2 = v$$

$$w^2 = 2e^{-\frac{\pi}{6}i}$$

$$w = \pm\left(2e^{-\frac{\pi}{6}i}\right)^{\frac{1}{2}}$$

$$= \pm\sqrt{2}\,e^{-\frac{\pi}{12}i}$$

$$\text{OR} \quad \sqrt{2}e^{-\frac{\pi}{12}i}, \quad \sqrt{2}e^{\frac{11\pi}{12}i}$$

<u>Note</u>: There are many other forms that could be used.

$$\pm\left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i\right), \quad \pm\left(\sqrt{\frac{2+\sqrt{3}}{2}} - \sqrt{\frac{2-\sqrt{3}}{2}}\,i\right) \quad \underline{\text{etc}}$$

Question 11 (continued)

(b) Evaluate $\int_1^2 x^2\sqrt{2-x}\,dx$. **3**

b) COMMENT:

Should have been a simple substitution early in the paper. Unfortunately some students made it much harder than it needed to be. Also, many made a mistake in expanding $(2-u)^2$ writing it as $4-2u+u^2$ and not $4-4u+u^2$.

$$I = \int_1^2 x^2\sqrt{2-x}\,dx$$

let $u = 2-x$

$\frac{du}{dx} = -1$

when $x = 1$ $\quad x = 2$

$u = 1$ $\quad u = 0$

$$I = \int_1^0 (2-u)^2\sqrt{u}\,.\,-du$$

$$= \int_0^1 u^{\frac{1}{2}}(4-4u+u^2)\,du$$

$$= \int_0^1 \left(4u^{\frac{1}{2}} - 4u^{3/2} + u^{5/2}\right)du$$

$$= \left[\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^1$$

$$= \frac{8}{3}(1)^{3/2} - \frac{8}{5}(1)^{5/2} + \frac{2}{7}(1)^{7/2} - (0)$$

$$= \frac{142}{105} \quad \text{OR} \quad 1\frac{37}{105}$$

Question 11 (continued)

(c) A quartic equation

$$iz^4 + (-3 - 7i)z^3 + (21 + 17i)z^2 + (-51 - 15i)z + 45 = 0$$

has 4 distinct roots, z_1, z_2, z_3, and z_4 which are represented by point A, B, C, and D respectively.
It is given $z_1 = -3i$, $z_3 = 3$, and $\text{Im}(z_4) > 0$

COMMENT:
It appeared as though students didn't know how to apply the sum and product of roots when the coefficients are complex. It also appeared that students were unfamiliar with naming quadrilaterals in order of its vertices.

Some students would benefit in asking themselves the question "Does my answer(s) seem reasonable?" Reasonable in the sense that they are not unnecessarily complicated, as there is no need for that (from an examiners point of view)

Question 11 (continued)

(c) (i) Find z_2 and z_4. **2**

$$z_1 + z_2 + z_3 + z_4 = \frac{-b}{a}$$

$$-3i + z_2 + 3 + z_4 = \frac{3+7i}{i} \times \frac{-i}{-i}$$

$$3 - \cancel{3i} + z_2 + z_4 = 7 - \cancel{3i}$$

$$z_2 + z_4 = 4 \quad \text{—— (1)}$$

$$z_1 . z_2 . z_3 . z_4 = \frac{e}{a}$$

$$(-3i) z_2 (3) z_4 = \frac{45}{i}$$

$$-9i . z_2 z_4 = \frac{45}{i}$$

$$z_2 z_4 = \frac{45}{i(-9i)}$$

$$z_2 z_4 = 5 \quad \text{—— (2)}$$

z_2 & z_4 are the roots of

$$x^2 - 4x + 5 = 0$$

$$x^2 - 4x + 4 = -1$$

$$(x-2)^2 = -1$$

$$x - 2 = \pm i$$

$$x = 2 \pm i$$

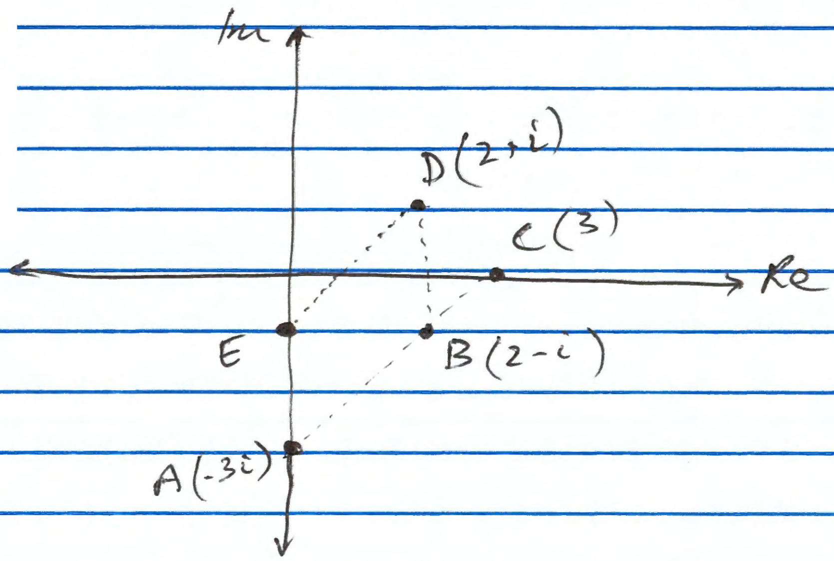
$z_2 = 2 - i$

$z_4 = 2 + i$ since $\text{Im}(z_4) > 0$

Question 11 (continued)

(c) (ii) Point E represents the complex number wz_3 such that $ABDE$ forms a parallelogram. **3**

By sketching the points A, B, and D on an Argand diagram, or otherwise, find w in the form $re^{i\theta}$ where $r > 0$, and $-\pi < \theta \leq \pi$.



ABDE is a parallelogram

$\therefore$ E corresponds to the complex number $-i$.

$$wz_3 = -i$$

$$w(3) = -i$$

$$w = \frac{-i}{3}$$

$$w = \frac{1}{3}(0 - i)$$

$$= \frac{1}{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{3}e^{-\frac{\pi}{2}i}$$

Question 11 (continued)

(d) Sketch on the one diagram of the Argand diagram defined by **3**

$$1 \le |z+2i| \le 2 \text{ and } \left|\arg(z+4i) - \frac{3\pi}{8}\right| \le \frac{\pi}{8}.$$

d) COMMENT:
most students struggled with the second condition.

Note: $\boxed{\begin{array}{c} |x-a| \le b \\ -b \le x-a \le b \end{array}}$ where $b>0$

$a-b \le x \le a+b$

The above result is helpful in finding the range of values that $\arg(z+4i)$ can take

$1 \le |z+2i| \le 2$

$1 \le |z-(-2i)| \le 2$

this can be understood as the distance from $(0,-2)$ is between 1 and 2.
which is the region between the concentric circles of radii 1 and 2 centred at $(0,-2)$

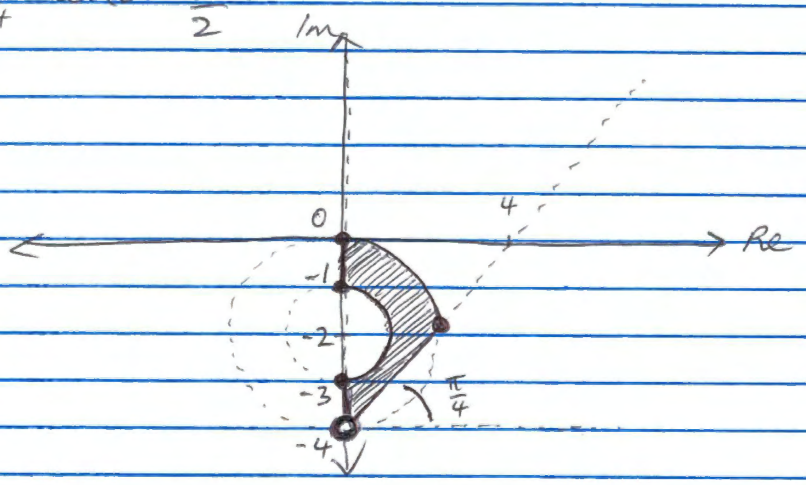
$\left|\arg(z+4i) - \frac{3\pi}{8}\right| \le \frac{\pi}{8}$

$-\frac{\pi}{8} \le \arg(z+4i) - \frac{3\pi}{8} \le \frac{\pi}{8}$

$\frac{\pi}{4} \le \arg(z+4i) \le \frac{\pi}{2}$

$\frac{\pi}{4} \le \arg(z-(-4i)) \le \frac{\pi}{2}$

this can be understood as the argument from the point $(0,-4)$ lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$.



End of Question 11 Solutions

Section II **Question 12 Solutions** **14 Marks**

(a) (i) Express $\dfrac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions. **3**

$$\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}=\frac{3x+7}{(x+1)(x+2)(x+3)}$$

$$\Rightarrow A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)=3x+7$$

$$A\left(x^2+5x+6\right)+B\left(x^2+4x+3\right)+C\left(x^2+3x+2\right)=3x+7$$

$A+B+C=0$ ①
$5A+4B+3C=3$ ②
$6A+3B+2C=7$ ③

② - 3×① $\Rightarrow 2A+B=3$ ④
③ - 2×① $\Rightarrow 4A+B=7$ ⑤

⑤ - ④ $\Rightarrow 2A=4$
$A=2$

sub back into ④ $\Rightarrow B=-1$
sub back into ① $\Rightarrow C=-1$

$$\frac{2}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}=\frac{3x+7}{(x+1)(x+2)(x+3)}$$

(ii) Hence prove that $\displaystyle\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx=\ln 2.$ **2**

$$\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}dx=\int_0^1\left(\frac{2}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}\right)dx$$

$$\begin{aligned}&=[2\ln|x+1|-\ln|x+2|-1n|x+3|]\\&=2\ln 2-\ln 3-\ln 4-(2\ln 1-\ln 2-\ln 3)\\&=2\ln 2-\ln 3-2\ln 2-0+\ln 2+\ln 3\\&=\ln 2\end{aligned}$$

Marker's Comments: This was done well by most pupils whether or not the method above was used or the 'cover up' method.
Some pupils did NO work and simply stated the constants for the numerators. For a three mark question, one must show all their working.
For pupils who got part i wrong, they needed to get the right solution for their equation in order to get full marks.

(b) Given that ω is one of the non-real roots of $z^3 = 1$, show that, $\dfrac{\omega^2}{1+\omega^2} = -\dfrac{1}{\omega^2}$. **2**

$$z^3 - 1 = (z-1)\left(z^2 + z + 1\right)$$

The solutions to the second product are all non-real since it's discriminant is less than zero.

$$\omega^2 + \omega + 1 = 0 \Rightarrow -\omega^2 = 1 + \omega$$

$$-\frac{1}{\omega^2} = \frac{1}{1+\omega} \times \frac{\omega^2}{\omega^2}$$

$$= \frac{\omega^2}{\omega^2 + \omega^3}$$

$$= \frac{\omega^2}{1+\omega^2}$$, regardless of which non-real root we choose.

Marker's Comments: This question was reasonably well done.
Many pupils used the actual non-real roots to prove this.
Most only used one of them and NOT the other. Both were needed if you chose to do this.

(c) A student builds a device designed to safely transport a particular fragile item that needs to be transported by dropping the item from a fixed height.

The total mass of the loaded device must be m kg and there is a parachute installed in the device which is designed to open after $\dfrac{1}{3k}$ seconds of motion.

When the parachute is opened the device experiences a resistance equal to mkv N, where k is a positive constant and v is its speed in metres per second.

It is observed that the loaded device reaches the ground and lands with a speed of $\dfrac{5g}{6k}$ m/s

Assume that without the parachute the air resistance of the device is negligible and that the acceleration due to gravity is g m/s^2.

(i) Using calculus, show that the speed of the loaded device at time $t = \dfrac{1}{3k}$ is $\dfrac{g}{3k}$ m/s. **2**

$\ddot{y} = -g, \dot{y} = -gt + C$, where C is the initial velocity.

Since the item is dropped from a fixed height, the initial velocity is 0 m/S.

Speed is $|\dot{y}|$.

$$\begin{aligned}\left|\dot{y}\left(\frac{1}{3k}\right)\right| &= \left|-g \times \frac{1}{3k}\right| \\ &= \frac{g}{3k}\end{aligned}$$

(ii) After the parachute has been opened show that $v = \frac{g}{k}\left(1 - \frac{2}{3}e^{\frac{1}{3}-kt}\right)$, **3**

for $t \geq \frac{1}{3k}$.

$$v = |\dot{y}|,\ v_0 = \frac{g}{3k} \text{ when } t = \frac{1}{3k}.$$

$$v = -\dot{y}, \text{ when } \dot{y} \leq 0.$$

$$\ddot{y} = -g + kv$$

$$\ddot{y} = -g - k\dot{y}$$

$$\int \frac{d\dot{y}}{g + k\dot{y}} = -\int dt$$

$$\frac{1}{k}\ln|g + k\dot{y}| = -t + C$$

$$\ln|g + k\dot{y}| = -kt + C'$$

$$g + k\dot{y} = Ae^{-kt} \quad ①$$

Consider initial conditions.

$$g + k\left(-\frac{g}{3k}\right) = Ae^{-k\frac{1}{3k}}$$

$$\frac{2g}{3}e^{\frac{1}{3}} = A \quad ②$$

Substitute ② into ①.

$$g + k\dot{y} = \frac{2}{3}ge^{\frac{1}{3}-kt}$$

$$\dot{y} = \frac{g}{k}\left(\frac{2}{3}e^{\frac{1}{3}-kt} - 1\right)$$

$$v = \frac{g}{k}\left(1 - \frac{2}{3}e^{\frac{1}{3}-kt}\right)$$

(iii) Find the time taken to reach the ground, in terms of k. **2**

$$v = \frac{5g}{6k} \quad \text{when it reaches the ground.}$$

$$\frac{5g}{6k} = \frac{g}{k}\left(1 - \frac{2}{3}e^{\frac{1}{3}-kt}\right)$$

$$\frac{5}{6} = 1 - \frac{2}{3}e^{\frac{1}{3}-kt}$$

$$\Rightarrow \frac{2}{3}e^{\frac{1}{3}-kt} = \frac{1}{6}$$

$$e^{\frac{1}{3}-kt} = \frac{1}{4}$$

$$\frac{1}{3} - kt = -2\ln 2$$

$$t = \frac{6\ln 2 + 1}{3k}$$

Marker's Comments: In general this question was done well with a few silly mistakes here and there. The intention of the question was to take the downward direction as positive, which makes most things simpler.

Part (i) was done very well.

Part (ii) was mostly done well.
Some algebra mistakes occurred in some working or occasionally a pupil used incorrect boundaries.

Part (iii) was generally done well.
Some pupils added time on at the end not realising the function given would give the exact time.
A number of students had algebra problems right at the very end.
Just for this part, this was mostly ignored.

Q13 SOLUTIONS 15 Marks

General feedback:

- Not enough care was taken to secure the full mark in one mark problems.
- On the flip side, responses to three and four mark problems were generally unnecessarily complicated.

~

A. A particle moves in a straight line with acceleration $a = -5e^{v}$.

The particle is initially at the origin with velocity 2 m/s.

I. Show that the particle comes to rest when $t = \frac{1}{5}\left(1 - e^{-2}\right)$. **2**

Solution	Comment(s)
Since $a = \frac{dv}{dt}$: $$\frac{dv}{dt} = -5e^{v}$$ $$\frac{dt}{dv} = \frac{1}{-5e^{v}}$$ The particle comes to rest at $v = 0$, so: $$t = \frac{1}{5}\int_{2}^{0} -e^{-v}\,dv$$ $$= \frac{1}{5}\left[e^{-v}\right]_{2}^{0}$$ $$= \frac{1}{5}\left(e^{0} - e^{-2}\right)$$ $$= \frac{1}{5}\left(1 - e^{-2}\right)$$	Students generally performed well on this question.

A II. Show that the particle stops when $x = \frac{1}{5}\left(1 - 3e^{-2}\right)$. **3**

<table>
<tr><td>Solution</td><td>Comment(s)</td></tr>
<tr><td>Since $a = v\frac{dv}{dx}$:
$$v\frac{dv}{dx} = -5e^v$$
$$\frac{dv}{dx} = \frac{-5e^v}{v}$$
$$\frac{dx}{dv} = \frac{v}{-5e^v}$$
The particle stops at $v = 0$, so:
$$x = \frac{1}{5}\int_2^0 -ve^{-v}\,dv$$
Using integration by parts:
$$U = v \qquad V' = -e^{-v}$$
$$U' = 1 \qquad V = e^{-v}$$
$$\begin{aligned} x &= \frac{1}{5}\left([ve^{-v}]_2^0 - \int_2^0 e^{-v}\,dv\right) \\ &= \frac{1}{5}(-2e^{-2} - [-e^{-v}]_2^0) \\ &= \frac{1}{5}\left(-2e^{-2} + (1 - e^{-2})\right) \\ &= \frac{1}{5}(1 - 3e^{-2}) \end{aligned}$$</td><td>Trying to solve this problem using $v = \frac{dx}{dt}$ is about as efficient as trying to solve $x^2 = 1$ using the quadratic formula.

Given the setup, using $a = v\frac{dv}{dx}$ instead is a much simpler approach.</td></tr>
</table>

III. Describe the motion. **1**

<table>
<tr><td>Solution</td><td>Comment(s)</td></tr>
<tr><td>Since $e^v > 0$, $a < 0$, so the acceleration of the particle is always decreasing.

Furthermore, the initial velocity is positive, so it will move while slowing down in the positive direction until it reaches the maximum displacement found in Part II at the time found in Part I.

After that, the particle will move in the negative direction while speeding up indefinitely.</td><td>Common error(s):
• Not mentioning the particle's movement in the negative direction after reaching its maximum displacement.
• Describing the motion as SHM.</td></tr>
</table>

B. A sequence $u_1, u_2, u_3 \ldots$ is such that for $n \in \mathbb{Z}^+$:

$$u_1 = \frac{1}{4}, \qquad u_{n+1} = u_n + \frac{1}{n(n+1)} + 2^{-n}$$

I. Prove by mathematical induction that for $n \in \mathbb{Z}^+$: **3**

$$u_n = \frac{9}{4} - \frac{1}{n} - 2^{-n+1}$$

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>
Base Case: Prove true for $n = 1$.
$$\begin{aligned} u_1 &= \frac{9}{4} - \frac{1}{1} - 2^{-1+1} \\ &= \frac{9}{4} - 1 - 2^0 \\ &= \frac{1}{4} \end{aligned}$$
Hence, true for $n = 1$.

Induction Hypothesis: Assume true for $n = k$.
$$u_k = \frac{9}{4} - \frac{1}{k} - 2^{-k+1}$$
Inductive Step: Prove true for $n = k + 1$, i.e. Prove that:
$$\begin{aligned} u_{k+1} &= \frac{9}{4} - \frac{1}{k+1} - 2^{-(k+1)+1} \\ &= \frac{9}{4} - \frac{1}{k+1} - 2^{-k} \end{aligned}$$
Using the given expression for u_{n+1} and the expression for u_k from the induction hypothesis:
$$\begin{aligned} u_{k+1} &= u_k + \frac{1}{k(k+1)} + 2^{-k} \\ &= \frac{9}{4} - \frac{1}{k} - 2^{-k+1} + \frac{1}{k(k+1)} + 2^{-k} \\ &= \frac{9}{4} - \left(\frac{1}{k} - \frac{1}{k(k+1)}\right) - \left((2 \times 2^{-k}) - 2^{-k}\right) \\ &= \frac{9}{4} - \left(\frac{k+1-1}{k(k+1)}\right) - 2^{-k}(2-1) \\ &= \frac{9}{4} - \frac{1}{k+1} - 2^{-k} \end{aligned}$$
Hence, by the principle of mathematical induction:
$$u_n = \frac{9}{4} - \frac{1}{n} - 2^{-n+1}$$
</td><td>Students generally performed well on this question.</td></tr>
</table>

B. II. Show that $u_n < \frac{9}{4}$ for $n \in \mathbb{Z}^+$. **1**

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>

From Part I:

$$\begin{aligned} u_n &= \frac{9}{4} - \frac{1}{n} - 2^{-n+1} \\ &= \frac{9}{4} - \left(\frac{1}{n} + 2^{-n+1}\right) \end{aligned}$$

Recognise that $\frac{1}{n} + 2^{-n+1} > 0$ for $n \in \mathbb{Z}^+$, so:

$$\begin{aligned} -\left(\frac{1}{n} + 2^{-n+1}\right) &< 0 \\ \frac{9}{4} - \left(\frac{1}{n} + 2^{-n+1}\right) &< \frac{9}{4} \\ \therefore u_n &< \frac{9}{4} \end{aligned}$$

</td><td>

Alternatively, as $n \to \infty$:

$$\begin{aligned} \frac{1}{n} + 2^{-n+1} &\to 0^+ \\ \frac{9}{4} - \left(\frac{1}{n} + 2^{-n+1}\right) &\to \left(\frac{9}{4}\right)^- \\ u_n &\to \left(\frac{9}{4}\right)^- \\ \therefore u_n &< \frac{9}{4} \end{aligned}$$

~

Common error(s):

- Not mentioning the direction that $\frac{1}{n} + 2^{-n+1}$ is approaching 0 from.

</td></tr>
</table>

C. The equation of motion for a particle is given by:

$$\frac{dv}{dt} = -n^2 x$$

where n is a positive constant, x is the displacement at time t and v is the velocity at time t.

I. Show that $v^2 = n^2(A^2 - x^2)$, where A is a constant, satisfies the above equation. **1**

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>

Since $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a = \frac{dv}{dt}$:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dx}\left(\frac{1}{2}n^2(A^2 - x^2)\right) \\ &= \frac{d}{dx}\left(\frac{1}{2}n^2A^2 - \frac{1}{2}n^2x^2\right) \\ &= \frac{-2}{2}n^2x \\ &= -n^2x \end{aligned}$$

</td><td>

Most students don't seem to understand that showing Equation A satisfies Equation B means that you can start from A.

This might explain the number of responses that tried to solve the problem using integration instead of differentiation.

Regardless, this is a concerning trend, considering that students were taught how to approach this in ME1 exponential growth and decay problems.

Common error(s):

- Arbitrarily assigning the constant of integration c to the constants A and n instead of using the fact that $x = A$ when $v = 0$.

</td></tr>
</table>

C. II. If the particle is initially at the origin, find the first time that the particle's speed is half its maximum speed. **4**

<table>
<tr><td>Solution</td><td>Comment(s)</td></tr>
<tr><td>Since $a = -n^2x$, the particle is undergoing simple harmonic motion.

As the particle is initially at the origin, its equation of motion can be modelled by $x = A\sin nt$.

Differentiating to find the velocity:
$$\dot{x} = An\cos nt$$
The amplitude of $\dot{x}$ is An, which is the maximum velocity, so:
$$\frac{An}{2} = An\cos nt$$
$$\cos nt = \frac{1}{2}$$
$$nt = \frac{\pi}{3} + 2k\pi \qquad nt = \frac{5\pi}{3} + 2k\pi$$
The first time that the particle's velocity is half of its maximum velocity occurs in the first quadrant of the first revolution, so:
$$nt = \frac{\pi}{3} + 2k\pi$$
$$k = 0$$
$$\therefore nt = \frac{\pi}{3}$$
$$t = \frac{\pi}{3n}$$</td><td>Alternatively, since $v^2 = n^2(A^2 - x^2)$ is a concave down parabola translated up by n^2A^2 units, its vertex is at $(0, n^2A^2)$.

Solving $v^2 = n^2A^2$ then gives the maximum velocity $v = An$, as $v = -An$ gives the minimum velocity.

The solution then proceeds as shown previously.

~

Although a fair portion of students received full marks, they tended to waste time by taking convoluted approaches.

Common error(s):
• Finding an expression for the displacement instead of time.</td></tr>
</table>

End of Question 13 Solutions

(a) A particle P is projected from the origin with initial speed V m/s at an angle 45° above the positive x-axis.

The position vector of the particle, $\mathbf{r}(t)$, where t is the time in seconds after the particle is projected, is given by

$$\mathbf{r}(t)=\begin{pmatrix} \dfrac{Vt}{\sqrt{2}} \\ \\ -\dfrac{1}{2}gt^2+\dfrac{Vt}{\sqrt{2}} \end{pmatrix}. \qquad \text{(Do NOT prove this)}$$

(i) Show that the equation of the trajectory of P is **2**

$$y = x - \frac{g}{V^2}x^2.$$

$$\mathbf{r}(t) = \begin{pmatrix} \frac{vt}{\sqrt{2}} \\ -\frac{1}{2}gt^2 + \frac{vt}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\therefore x = \frac{vt}{\sqrt{2}} \Rightarrow t = \frac{\sqrt{2}x}{v} \quad (1)$$

$$y = -\frac{1}{2}gt^2 + \frac{vt}{\sqrt{2}} \quad (2)$$

Sub (1) into (2)

$$y = -\frac{1}{2}g\left(\frac{\sqrt{2}x}{v}\right)^2 + \frac{v\left(\frac{\sqrt{2}x}{v}\right)}{\sqrt{2}}$$

$$= -\frac{1}{2}g\left(\frac{2x^2}{v^2}\right) + \frac{v}{\sqrt{2}}\left(\frac{\sqrt{2}}{v}x\right)$$

$$= -\frac{g}{v^2}x^2 + x$$

$$y = x - \frac{g}{v^2}x^2$$

Marking Scheme	Marker's comments
2 marks – Correct solution **1 mark** – Obtaining $t=\dfrac{x\sqrt{2}}{v}$	- This question was done well by majority of the candidates.

Question 14 (continued)

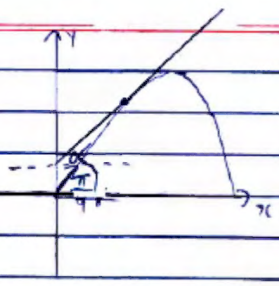
(a) (continued)

The point of projection (the origin) is on the floor of a barn.

The roof of the barn is given by the equation $y = x \tan \alpha + b$, where $b > 0$, and α is an acute angle.

(ii) Show that, if the particle just touches the roof then **2**

$$V(-1+\tan\alpha) = -2\sqrt{bg}.$$



$y = x - \frac{g}{V^2}x^2$ (From (a)(i))

$y = x\tan\alpha + b$

$\therefore x - \frac{g}{V^2}x^2 = x\tan\alpha + b$ (to find where the trajectory and roof meet)

$\frac{g}{V^2}x^2 + x\tan\alpha - x + b = 0$

$\therefore \frac{g}{V^2}x^2 + x(\tan\alpha - 1) + b = 0$ $(*)$

If the particle just touches the roof

$\therefore \Delta = 0$

$\therefore (\tan\alpha - 1)^2 - 4\times\left(\frac{g}{V^2}\right)\times b = 0$

$(\tan\alpha - 1)^2 = \frac{4gb}{V^2}$

$\tan\alpha - 1 = \pm\frac{2\sqrt{bg}}{V}$

Since projection angle is $45°$

$\alpha < \frac{\pi}{4}$ (Otherwice projectile wont touch roof)

$\therefore \tan\alpha < \tan\frac{\pi}{4}$

$\therefore \tan\alpha < 1 \rightarrow \tan\alpha - 1 < 0$

and $V > 0$ (Speed)

$\therefore \tan\alpha - 1 = -\frac{2\sqrt{bg}}{V}$

$\therefore V(-1+\tan\alpha) = -2\sqrt{bg}$

Question 14 (continued)

(a) (ii) (continued)

Marking Scheme	Marker's comments
2 marks – Correct solution **1 mark** – Correct working to obtain $$\frac{g}{v^2}x^2 + x\left(\tan\alpha - 1\right) + b = 0.$$	- This question was not done well by many candidates. - Many candidates incorrectly assumed that the slope of the roof would touch the trajectory of the particle at the vertex, which did not lead to the desired result. - A significant number of candidates were penalised for not correctly justifying or even specifically addressing when $\tan\alpha - 1 = \pm\frac{2\sqrt{bg}}{v}$, why $\tan\alpha - 1 < 0$ in relation to the context of the question. Candidates need to refer to solution for proper reasoning. - Candidates who stated α is acute and hence why $\tan\alpha - 1 < 0$ is not entirely correct as when $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, $\tan\alpha - 1 > 0$. - Some candidates were incorrectly manipulating their work to achieve the desired result. Candidates should ensure that if their solution does not yield the correct result, they should review their work rather than "fudge" their working.

Question 14 (continued)

(a) (continued)

(iii) If this condition is satisfied, find, in terms of α, V, and g, the time after projection at which touching takes place. **2**

Using the result (*) in (a)(ii)

$$\frac{g}{V^2}x^2 + x(\tan\alpha - 1) + h = 0$$

and $\Delta = (\tan - 1)^2 - \frac{4gh}{V^2} = 0$

$$x = \frac{-(\tan\alpha - 1) \pm \sqrt{(\tan\alpha - 1)^2 - \frac{4gh}{V^2}}}{2\left(\frac{g}{V^2}\right)}$$ (quadratic Formula)

$$x = \frac{(1-\tan\alpha)V^2}{2g} \quad (1)$$

Also $x = \frac{V(t)}{\sqrt{2}} \Rightarrow t = \frac{\sqrt{2}x}{V} \quad (2)$

Sub (1) into (2)

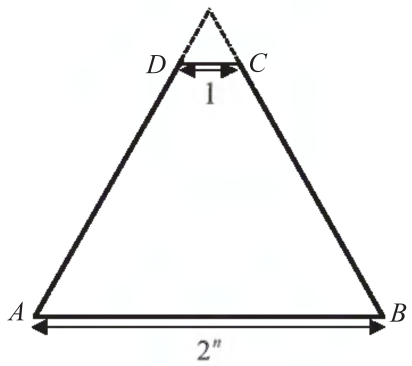
$$t = \frac{\sqrt{2}\,\frac{(1-\tan\alpha)V^2}{2g}}{V}$$

$$= \frac{\sqrt{2}V(1-\tan\alpha)}{2g} \text{ seconds}$$

Marking Scheme	Marker's comments
2 marks – Correct solution **1 mark** – Substantial progress to correct solution.	- Many candidates were not successful in this question, particularly if they were not able to achieve the result in (ii).

Question 14 (continued)

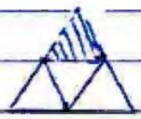
(b) A shape $ABCD$ is formed by taking an equilateral triangle of side length 2^n (n is a positive integer) and removing an equilateral triangle of side length 1 from one of its corners.



A trapezium tile, see diagram below, consists of three equilateral triangles of side length 1 unit.

(i) Prove using induction that trapezium tiles, can fully cover the shape without any overlapping for all positive integers n. **3**

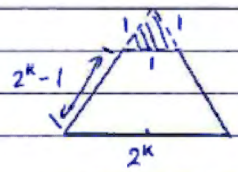
Test for $n=1$



1 trapezium tile covers the shape.
$\therefore$ True for $n=1$

Assume the statement is true for $n=k$, $k \in Z^+$

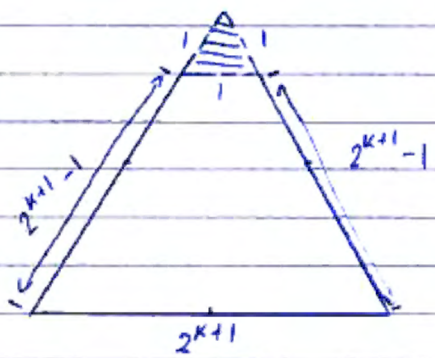
i.e.



Trapezium tiles can cover the shape.

Hence to prove true for $n=k+1$

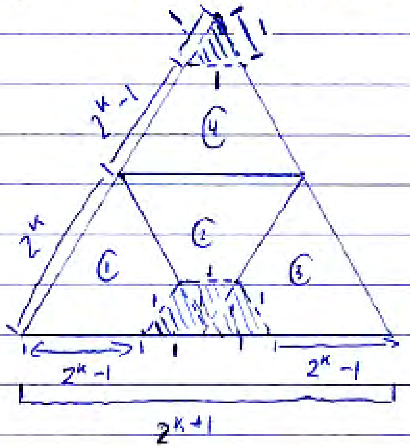
RTP: Trapezium tiles can cover the shape below



Question 14 (continued)

(b) (i) (continued)

An equilateral triangle with side length 2^{k+1} can be made up of 4 equilateral triangles with side length 2^k as shown below.



As the equilateral triangle above can be filled with 4 2^k triangles which are assumed true to be filled with the trapezium tiles, the gap shown from the configuration above can be filled with an additional trapezium tile.

$\therefore$ True for $n=k+1$

$\therefore$ The statement is proven true by Mathematical Induction for all positive integers n.

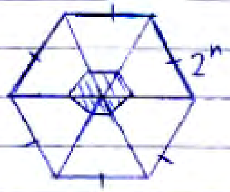
Marking Scheme	Marker's comments
3 marks – Correct proof **2 marks** – Shows substantial relevant progress in proving the case for $n = k+1$. **1 mark** – Establish the result for $n=1$ and include a diagram.	- Many candidates were not successful in proving the $n = k + 1$ case. Common issues included failing to properly utilise the assumption in their proof, relying on excessive and irrelevant statements or diagrams without providing a clear and logical argument, or demonstrating a lack of understanding of how to approach the proof. - Candidates are encouraged to use diagrams as part of their proof, especially with Mathematical Induction involving geometry.

Question 14 (continued)

(b) **(continued)**

(ii) Hence explain why a regular hexagon with side length 2^n can be fully covered by these trapezium tiles, <u>and</u> determine, in terms of n, the number of tiles required. **3**

A hexagon with side length 2^n is made up of 6 equilateral triangles with side length 2^n



This leaves the middle blank which can be filled with 2 trapezium tiles.

From the previous question in b(i), each "triangle" (2^{n+1}) is made up of 4 smaller triangles (2^n) and 1 trapezium tile.

let S_n be the amount of trapezium tiles used in constructing triangles of side length 2^n

$\therefore S_1 = 1$

$S_2 = 4 \times S_1 + 1 = 4 \times 1 + 1 = 5$

$S_3 = 4 \times S_2 + 1 = 4 \times (4 \times 1 + 1) + 1 = 4^2 + 4 + 1 = 21$

$S_4 = 4 \times S_3 + 1 = 4 \times (4^2 + 4 + 1) + 1 = 4^3 + 4^2 + 4 + 1 = 85$

$\vdots$

$S_n = 4 \times S_{n-1} + 1 = 4^{n-1} + 4^{n-2} + \dots + 4^2 + 4 + 1$

$= 1 + 4 + 4^2 + \dots + 4^{n-2} + 4^{n-1}$

This is a GP with $a = 1$, $r = 4$, $n = n$

$\therefore S_n = \dfrac{1(4^n - 1)}{4 - 1}$

$= \dfrac{4^n - 1}{3}$

$\therefore$ The hexagon will have $6 \times S_n + 2 = 6\left(\dfrac{4^n - 1}{3}\right) + 2$

$= 2 \times 4^n - 2 + 2$

$= 2 \times 4^n = 2^{2n+1}$

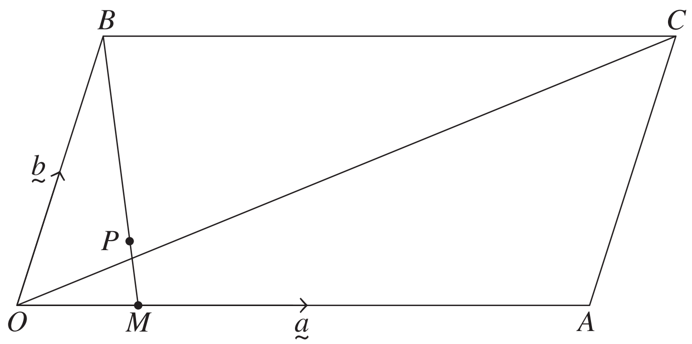
Question 14 (continued)

(b) (ii) (continued)

Marking Scheme	Marker's comments
3 marks – Correct explanation of why regular hexagon with side length of 2^n can be fully covered by trapezium tiles AND correct solution in finding n. **2 marks** – Correct explanation and shows some relevant progress in finding n or equivalent merit. **1 mark** – Correct explanation or shows some relevant progress in finding n.	- This question was not done well by many candidates. - Some candidates struggled to explain why regular hexagon with side length of 2^n can be fully covered by trapezium tiles especially when they struggled in proving in part (i). - Significant number of candidates were not successful in finding n with some not considering that a single trapezium consists of 3 equilateral triangles of side length 1 unit or silly errors. Others simply struggled to find the correct solution.

Question 14 (continued)

(c) Let $OACB$ be a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$
M is a point on OA such that $OM = \frac{1}{5}OA$.
P is a point on MB such that $MP = \frac{1}{6}MB$, as shown below.



(i) Show that P lies on OC. **3**

$\overrightarrow{BC} = \overrightarrow{OA}$ (opposite sides of parallelogram are equal)

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$
$= \underset{\sim}{b} + \underset{\sim}{a}$

$\overrightarrow{OM} = \frac{1}{5}\overrightarrow{OA} = \frac{1}{5}\underset{\sim}{a}$

$\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB}$
$= -\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{MP} = \frac{1}{6}\overrightarrow{MB} = \frac{1}{6}(-\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{b}) = -\frac{1}{30}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$

$\therefore \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$
$= \frac{1}{5}\underset{\sim}{a} + (-\frac{1}{30}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b})$

$= \frac{1}{6}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$

$= \frac{1}{6}(\underset{\sim}{a} + \underset{\sim}{b})$
$= \frac{1}{6}\overrightarrow{OC}$

Since $\overrightarrow{OP} = \lambda\overrightarrow{OC}$ where $\lambda \in \mathbb{R}$
$\therefore \overrightarrow{OP} \parallel \overrightarrow{OC}$ with a common point O
$\therefore O, P$ and C are collinear
$\therefore P$ lies on $\overrightarrow{OC}$

Question 14 (continued)

(c) (i) (continued)

Marking Scheme	Marker's comments
3 marks – Complete solution with working and justification. **2 marks** – Significant relevant progress **1 mark** – Obtaining $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$	- This question was generally answered well by candidates. - Candidates should give brief explanation why if $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$ or equivalent merit, then *P* lies on *OC*.

(ii) State the ratio of length *OP* : *PC*. **1**

$\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$

$\therefore \overrightarrow{PC} = \frac{5}{6}\overrightarrow{OC}$

$\therefore |\overrightarrow{OP}| : |\overrightarrow{PC}| = 1:5$

Marking Scheme	Marker's comments
1 mark – Correct answer	- A significant number of candidates wrote the incorrect answer of 1 : 6, possibly making the silly error of mistaking *OC* for *PC*.

Question 15 Solutions — 16 marks

(a) A hose situated at the origin sprays water with initial speed u m/s and at angle of θ to the positive x-axis.

The hose sprays water to a height of 80 m.

We will consider the motion of one water particle in the spray.

(i) Assuming no air resistance, the position vector of the particle, $\underset{\sim}{r}(t)$, where t is the time in seconds after the water starts to spray, is given by **2**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ \\ -5t^2 + ut\sin\theta \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

Show that $u \sin\theta = 40$.

$$y = -5t^2 + ut\sin\theta$$
$$\dot{y} = -10t + u\sin\theta$$

Maximum height when $\dot{y} = 0$

$$\dot{y} = 0 \Rightarrow t = \frac{u\sin\theta}{10}$$

$$\therefore y_{\text{MAX}} = -5\left(\frac{u\sin\theta}{10}\right)^2 + u\left(\frac{u\sin\theta}{10}\right)\sin\theta$$
$$= -\frac{u^2\sin^2\theta}{20} + \frac{u^2\sin^2\theta}{10}$$

$$= \frac{u^2\sin^2\theta}{20}$$

$$\therefore \frac{u^2\sin^2\theta}{20} = 80 \Rightarrow u^2\sin^2\theta = 1600$$

$$\therefore u^2\sin^2\theta = 40 \qquad (u\sin\theta > 0)$$

Comment: This was generally done well.

(a) (continued)

Taking into account air resistance, the acceleration, $\underset{\sim}{a}$, of a water drop is given by

$$\underset{\sim}{a}(t)=\begin{pmatrix}-0.2\dot{x}\\ \\ -10-0.2\dot{y}\end{pmatrix},$$

where x and y refers to the horizontal and vertical displacement respectively of a water drop at time t seconds.

(ii) Show that $y=-5(50+u\sin\theta)e^{-0.2t}-50t+5(50+u\sin\theta)$. **4**

Method 1:

$$\frac{dv}{dt}=-10-\frac{1}{5}v$$

$$=-\frac{50+v}{5}$$

$$\therefore \frac{1}{50+v}\,dv=-\frac{1}{5}\,dt$$

Note as $v > 0$ then $50 + v > 0$

$t=0,\ v=u\sin\theta$

$$\therefore \int_{u\sin\theta}^{v}\frac{1}{50+V}\,dV=-\frac{1}{5}\int_{0}^{t}\,dt$$

$$\therefore \left[\ln(50+V)\right]_{u\sin\theta}^{v}=-\frac{1}{5}(t-0)$$

$$\therefore \ln(50+v)-\ln(50+u\sin\theta)=-\frac{1}{5}t$$

$$\therefore \ln\left(\frac{50+v}{50+u\sin\theta}\right)=-0.2t$$

$$\therefore \frac{50+v}{50+u\sin\theta}=e^{-0.2t}$$

$$\therefore 50+v=(50+u\sin\theta)e^{-0.2t}$$

$$\therefore v=(50+u\sin\theta)e^{-0.2t}-50$$

(a) (ii) **Method 1** (continued)

$$\frac{dy}{dt} = (50 + u\sin\theta)e^{-0.2t} - 50$$
$$t = 0, y = 0$$

$$\therefore \int_0^y dY = \int_0^t (50 + u\sin\theta)e^{-0.2T} - 50\, dT$$

$$\therefore y - 0 = \left[-5(50 + u\sin\theta)e^{-0.2T} - 50T\right]_0^t$$

$$\therefore y = -5(50 + u\sin\theta)e^{-0.2t} - 50t - \left[-5(50 + u\sin\theta) - 0\right]$$

$$\therefore y = -5(50 + u\sin\theta)e^{-0.2t} - 50t + 5(50 + u\sin\theta)$$

Method 2: Same start as Method 1

$$\therefore \int \frac{1}{50+v}\, dv = -\frac{1}{5}\int dt$$

$$\therefore \ln(50+v) = -\frac{1}{5}t + C \qquad [t = 0, v = 50 + u\sin\theta]$$

$$\therefore C = \ln(50+v)$$

$$\therefore \ln(50+v) - \ln(50 + u\sin\theta) = -\frac{1}{5}t$$

$$\therefore \ln\left(\frac{50+v}{50+u\sin\theta}\right) = -0.2t$$

$$\therefore \frac{50+v}{50+u\sin\theta} = e^{-0.2t}$$

$$\therefore 50 + v = (50 + u\sin\theta)e^{-0.2t}$$

$$\therefore v = (50 + u\sin\theta)e^{-0.2t} - 50$$

(a) (ii) **Method 2** (continued)

Now $\frac{dy}{dt} = (50 + u\sin\theta)e^{-0.2t} - 50$

$$\therefore \int dy = \int_0 (50 + u\sin\theta)e^{-0.2t} - 50 \, dt$$

$$\therefore y + C_1 = -5(50 + u\sin\theta)e^{-0.2t} - 50t \qquad [t = 0, y = 0]$$

$$\therefore C_1 = -5(50 + u\sin\theta)$$

$$\therefore y = -5(50 + u\sin\theta)e^{-0.2t} - 50t - C_1$$

$$\therefore y = -5(50 + u\sin\theta)e^{-0.2t} - 50t + 5(50 + u\sin\theta)$$

Comment: Many students' work is unreadable and the setting out is hard to follow. It is not up to the marker to try and work out the students' logic.

You must present your arguments clearly

Students should simplify at each step before continuing.

Some students obviously didn't read the question properly and started on the horizontal components of the motion.

Students need to put in all the steps, but many found this hard with their bad setting out. Bad handwriting is one thing, but atrocious setting out takes all this to another level.

(a) (continued)

(iii)	Hence show that $y = 5\left[-\dot{y} + u\sin\theta + 50\ln\left(\dfrac{10+0.2\dot{y}}{10+0.2u\sin\theta}\right)\right]$	**3**

From (a) (ii):

(1): $\dot{y} + 50 = (50 + u\sin\alpha)\, e^{-0.2t}$

(2): $e^{-0.2t} = \dfrac{50+\dot{y}}{50+u\sin\alpha}$ and so

$$-0.2t = \ln\left(\frac{50+\dot{y}}{50+u\sin\alpha}\right)$$

(3): $t = -5\ln\left(\dfrac{10+0.2\dot{y}}{10+0.2u\sin\alpha}\right)$ $\quad[\div 5]$

Also (4): $y = -5(50+u\sin\alpha)\,e^{-0.2t} - 50t + 5(50+u\sin\alpha)$

Substitute (1) & (3) into (4):

$$y = -5(\dot{y}+50) - 50\times\left(-5\ln\left(\frac{10+0.2\dot{y}}{10+0.2u\sin\alpha}\right)\right) + 5(50+u\sin\alpha)$$

$$= 5\left(-\dot{y} + 50\ln\left(\frac{10+0.2\dot{y}}{10+0.2u\sin\alpha}\right) + u\sin\alpha\right) - 250 + 250$$

$$= 5\left(-\dot{y} + 50\ln\left(\frac{10+0.2\dot{y}}{10+0.2u\sin\alpha}\right) + u\sin\alpha\right)$$

Comment: This is a ‘Show that’ and also “Hence” question.
This is not the first time students have had such questions.
Hopefully, the concept will sink in before the HSC.

Students need to put in all the steps, but many found this hard with their bad setting out.
Bad handwriting is one thing, but atrocious setting out takes all this to another level.

Students who opted for an alternative solution that did not rely on ‘Hence’ could not get full marks.

(a) (continued)

(iv)	How high does the water from the hose reach now?	**1**

$\dot{y} = 0$ for the maximum height and $u \sin \alpha = 40$

Using (iii):

$$H_{max} = \frac{1}{0.2}\left(40 + \frac{10}{0.2}\ln\frac{\qquad}{40}\right)$$

$$= 53\text{m}$$

Comment: Students had to write down a numerical result in order to get full marks.
This was the whole reason why part (i) was asked.

Students were not penalised for an 'exact' answer, but they do have to realise that when you are giving the height of something like this, the best practice is to say the approximate value. If confused, students should write down the exact value and then the approximation.

(b) Suppose f and g are real-valued continuous functions defined on $[0, a]$, where $a > 0$, satisfy the conditions:

$$f(x) = f(a - x) \text{ and}$$

$$g(x) + g(a - x) = m, \text{ where } m \in \mathbb{R}.$$

(i) Show that $\displaystyle\int_0^a f(x)\,g(x)\ dx = \frac{m}{2}\int_0^a f(x)\ dx$. **3**

$$\int_0^a \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\left[m - \mathrm{g}(a-x)\right]\mathrm{d}x \qquad \text{Let } y = a - x,\ \frac{\mathrm{d}y}{\mathrm{d}x} = -1$$

$$= \int_a^0 \mathrm{f}(y)\left[m - \mathrm{g}(y)\right](-1)\,\mathrm{d}y$$

$$= \int_0^a \mathrm{f}(y)\left[m - \mathrm{g}(y)\right]\mathrm{d}y$$

$$= m\int_0^a \mathrm{f}(x)\,\mathrm{d}x - \int_0^a \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x$$

$$\therefore 2\int_0^a \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x = m\int_0^a \mathrm{f}(x)\,\mathrm{d}x$$

$$\int_0^a \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x = \frac{m}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x$$

Comment: Students cannot assume the result that many call 'King's Rule'.
They have to prove it if they want to use it.
This was made clear in the Task 3 Feedback.

As a result these students could only get a maximum of 2 marks.

From reading time, students would have needed to prove this in Question 16, but they decided to just assume it here.

Students need to put in all the steps, but many found this hard with their bad setting out. Bad handwriting is one thing, but atrocious setting out takes all this to another level.

(b) (continued)

(ii) Hence evaluate $\displaystyle\int_0^{\pi} x \sin x \cos^4 x \, dx$. **3**

Let $\mathrm{f}(x) = \sin x \cos^4 x$ and $\mathrm{g}(x) = x$,

then we have $\mathrm{f}(\pi - x) = \sin(\pi - x)\cos^4(\pi - x)$

$$= (\sin x)[-\cos x]^4 = \mathrm{f}(x)$$

and $g(x) + g(\pi - x) = x + (\pi - x) = \pi = m$

Hence, $\displaystyle\int_0^{\pi} x \sin x \cos^4 x \, \mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \sin x \cos^4 x \, \mathrm{d}x$

$$= \frac{\pi}{2}\left[-\frac{\cos^5 x}{5}\right]_0^{\pi} = \frac{\pi}{2}\left[-\left(\frac{-1}{5} - \frac{1}{5}\right)\right] = \frac{\pi}{5}$$

Comment: This question became much harder than the writer obviously intended.

Students could only gain marks if they had the right f, g and m, and especially $m \in \mathbb{R}$. As a result many students scored 0 on this question.

Students were not penalised for not justifying why the f and g above worked, but if they didn't then they usually fell into the trap below.

Some students had the correct numerical answer, but their logic/working was flawed. They made the problem into something that they knew could work out, WITHOUT any justification.
As a result these students were not successful in gaining any marks.

Students need to put in all the steps, but many found this hard with their bad setting out. Bad handwriting is one thing, but atrocious setting out takes all this to another level.

End of Question 15 Solutions

(a) Given $I_n = \int_0^1 (1-x)^n e^x \, dx$, where n is a non-negative integer.

(a) COMMENT:
It is important to show all steps with a prove or show that question.
Students should not expect full marks for writing one line

$$I_n = \left[e^x (1-x)^n\right]_0^1 + n\int_0^1 (1-x)^{n-1} e^x dx$$

Many students struggled to use the reduction formula in part (ii)

(i) Show that $I_n = -1 + n\, I_{n-1}$ for $n \geq 1$. **2**

i) $I_n = \int_0^1 (1-x)^n e^x dx$

$u = (1-x)^n \qquad v' = e^x$

$u' = n(1-x)^{n-1} \cdot -1 \qquad v = e^x$

$$I_n = \left[(1-x)^n e^x\right]_0^1 + n\int_0^1 (1-x)^{n-1} e^x dx$$

$= \cancel{(1-1)^n e^1} - (1-0)^n e^0 + n I_{n-1}$

$= -1 + n I_{n-1}$

Question 16 (continued)

(a) (ii) Evaluate $\int_0^1 (1-x)^3 e^x \, dx$. **2**

$\int_0^1 (1-x)^3 e^x \, dx$

$= I_3$

$= -1 + 3.I_2$

$= -1 + 3(-1 + 2.I_1)$

$= -1 - 3 + 6 I_1$

$= -4 + 6(-1 + 1.I_0)$

$= -4 - 6 + 6 I_0$

$= -10 + 6 \int_0^1 e^x \, dx$

$= -10 + 6 \left[e^x \right]_0^1$

$= -10 + 6 \left[e^1 - e^0 \right]$

$= -10 + 6e - 6$

$= -16 + 6e$

b) COMMENT:

A lot of students failed to appropriately consider the absolute value.

A graph should have been considered at some point.

Clearly $f(x) = x|\cos x| \geq 0$ for $[0, 2\pi]$

and so $\int_0^{2\pi} f(x)$

(i) Prove that $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$, for constant a. **1**

$LHS = \int_0^a f(a-x)\,dx$

let $u = a - x$

$\frac{du}{dx} = -1$

$dx = -du$

when $x = 0$ $\quad$ $x = a$

$u = a$ $\quad$ $u = 0$

$= \int_a^0 f(u) . -du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

$= RHS$

OR $y = f(a-x)$ is the reflection of $y = f(x)$ about the line $x = \frac{a}{2}$

And so the signed area between the curve $y = f(a-x)$ and the x-axis is equal to that between $y = f(x)$ and the x-axis for $[0, a]$

$\therefore \int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$

Question 16 (continued)

(b) (ii) Hence or otherwise, evaluate $\int_0^{2\pi} x\,|\cos x|\,dx$. **3**

Hence

Let $I = \int_0^{2\pi} x|\cos x|\,dx$

$I = \int_0^{2\pi} (2\pi - x)|\cos(2\pi - x)|\,dx$

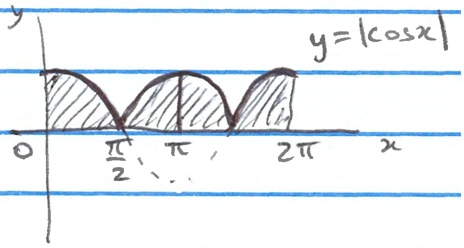
$I = \int_0^{2\pi} (2\pi - x)|\cos x|\,dx$

$I = 2\pi\int_0^{2\pi} |\cos x|\,dx - \int_0^{2\pi} x|\cos x|\,dx$

$I = 2\pi\int_0^{2\pi} |\cos x|\,dx - I$

$2I = 2\pi\int_0^{2\pi} |\cos x|\,dx$

$I = \pi\int_0^{2\pi} |\cos x|\,dx$



$I = 4\pi\int_0^{\frac{\pi}{2}} \cos x\,dx$

$I = 4\pi\left[\sin x\right]_0^{\frac{\pi}{2}}$

$I = 4\pi\left[\sin\frac{\pi}{2} - \cancel{\sin 0}\right]$

$I = 4\pi$.

Question 16 (continued)

(b) (ii) (continued)

OR (otherwise)

$$I = \int_0^{2\pi} x|\cos x|\,dx = \int_0^{\frac{\pi}{2}} x|\cos x|\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x|\cos x|\,dx + \int_{\frac{3\pi}{2}}^{2\pi} x|\cos x|\,dx$$

$$= \int_0^{\frac{\pi}{2}} x\cos x\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x(-\cos x)\,dx + \int_{\frac{3\pi}{2}}^{2\pi} x\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} x\cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos x\,dx + \int_{\frac{3\pi}{2}}^{2\pi} x\cos x\,dx$$

Let $J = \int x\cos x\,dx$ $\quad u = x \quad v' = \cos x$, $\quad u' = 1 \quad v = \sin x$

$$J = x\sin x - \int \sin x\,dx$$

$$J = x\sin x + \cos x + C$$

$$I = \Big[x\sin x + \cos x\Big]_0^{\frac{\pi}{2}} - \Big[x\sin x + \cos x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \Big[x\sin x + \cos x\Big]_{\frac{3\pi}{2}}^{2\pi}$$

$$= \frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - (0.\sin 0 + \cos 0) - \left[\frac{3\pi}{2}\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} - \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)\right]$$

$$+ 2\pi\sin 2\pi + \cos 2\pi - \left(\frac{3\pi}{2}\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2}\right)$$

$$= \frac{\pi}{2}(1) + (0) - (0 + 1) - \left[\frac{3\pi}{2}(-1) + (0) - \left(\frac{\pi}{2}(1) + (0)\right)\right] + 2\pi(0) + (1) - \left(\frac{3\pi}{2}(-1) + (0)\right)$$

$$= \frac{\pi}{2} - \cancel{1} + \frac{3\pi}{2} + \frac{\pi}{2} + \cancel{1} + \frac{3\pi}{2}$$

$$= 4\pi$$

(c) Let n be an integer, such that $n \neq 1$.

(i) Prove that $\sin\frac{\pi}{2n}\sum_{k=1}^{n-1}\sin\frac{k\pi}{n} = \cos\frac{\pi}{2n}$ **2**

$$LHS = \sin\frac{\pi}{2n}\sum_{k=1}^{n-1}\sin\frac{k\pi}{n}$$

$$= \sum_{k=1}^{n-1}\sin\frac{k\pi}{n}.\sin\frac{\pi}{2n}$$

$$= \sum_{k=1}^{n-1}\frac{1}{2}\left(\cos\left(\frac{k\pi}{n}-\frac{\pi}{2n}\right)-\cos\left(\frac{k\pi}{n}+\frac{\pi}{2n}\right)\right)$$

$$= \frac{1}{2}\sum_{k=1}^{n-1}\left(\cos\frac{(2k-1)\pi}{2n} - \cos\left(\frac{(2k+1)\pi}{2n}\right)\right)$$

$$= \frac{1}{2}\left[\cos\frac{(2(1)-1)\pi}{2n} - \cos\frac{(2(1)+1)\pi}{2n} + \cos\left(\frac{2(2)-1)\pi}{2n}\right) - \cos\left(\frac{2(2)+1)\pi}{2n}\right) + \dots \right.$$

$$\left. + \cos\left(\frac{2(n-1)-1)\pi}{2n}\right) - \cos\left(\frac{2(n-1)+1)\pi}{2n}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} - \cancel{\cos\frac{3\pi}{2n}} + \cancel{\cos\frac{3\pi}{2n}} - \cancel{\cos\frac{5\pi}{2n}} + \dots + \cancel{\cos\frac{(2n-3)\pi}{2n}} - \cos\frac{(2n-1)\pi}{2n}\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} - \cos\left(\frac{2n\pi}{2n} - \frac{\pi}{2n}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} - \cos\left(\pi - \frac{\pi}{2n}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} - -\cos\left(\frac{\pi}{2n}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} + \cos\frac{\pi}{2n}\right]$$

$$= \cos\frac{\pi}{2n}$$

$$= RHS$$

Question 16 (continued)

(c) (i) (continued)

c) i) COMMENT:

Moving the constant $\sin\frac{\pi}{2n}$ into the summation means that one application of products to sums or differences can be done.

Given that the result we are working towards is a single term we should expect most terms of the (telescoping) series to cancel.

Question 16 (continued)

(c) (ii) Let $\alpha = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$. **4**

Prove that $\displaystyle\sum_{k=1}^{n-1}\left|\alpha^k - 1\right| = 2\cot\frac{\pi}{2n}$.

$$\left|\alpha^k - 1\right| = \left|\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k - 1\right|$$

$$= \left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} - 1\right|$$

$$= \left|\cancel{1} - 2\sin^2\frac{k\pi}{n} + i.2\sin\frac{k\pi}{n}\cos\frac{k\pi}{n} - \cancel{1}\right|$$

$$= \left|2i\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)\right|$$

$$= \left|2i\sin\frac{k\pi}{n}\right|\left|\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right|$$

$$= 2\left|\sin\frac{k\pi}{n}\right|.1$$

$*$ since $k \in \mathbb{Z}^+$ such that $1 \le k \le n-1$

$0 < \frac{k\pi}{n} < \pi$ ie $\sin\frac{k\pi}{n} > 0$

$$= 2\sin\frac{k\pi}{n}.$$

$$\therefore \sum_{k=1}^{n-1}\left|\alpha^k - 1\right| = \sum_{k=1}^{n-1} 2\sin\frac{k\pi}{n}$$

$$= 2\sum_{k=1}^{n-1}\sin\frac{k\pi}{n}$$

$$= 2.\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$$ using part (i)

$$= 2\cot\frac{\pi}{2n}$$

Question 16 (continued)

(c) (ii) (continued)

$|\alpha^k - 1|$ could be simplified in other ways:

$$|\alpha^k - 1| = \left|\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k - 1\right|$$

$$= \left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} - 1\right|$$

$$= \left|\left(\cos\frac{2k\pi}{n} - 1\right) + i\sin\frac{2k\pi}{n}\right|$$

$$= \sqrt{\left(\cos\frac{2k\pi}{n} - 1\right)^2 + \left(\sin\frac{2k\pi}{n}\right)^2}$$

$$= \sqrt{\cos^2\frac{2k\pi}{n} - 2\cos\frac{2k\pi}{n} + 1 + \sin^2\frac{2k\pi}{n}}$$

$$= \sqrt{\sin^2\frac{2k\pi}{n} + \cos^2\frac{2k\pi}{n} + 1 - 2\cos\frac{2k\pi}{n}}$$

$$= \sqrt{1 + 1 - 2\cos\frac{2k\pi}{n}}$$

$$= \sqrt{2\left(1 - \cos\frac{2k\pi}{n}\right)}$$

$$= \sqrt{2\left(1 - \left(1 - 2\sin^2\frac{k\pi}{n}\right)\right)}$$

$$= \sqrt{4\sin^2\frac{k\pi}{n}}$$

$$= 2\left|\sin\frac{k\pi}{n}\right| \quad *$$

$$= 2\sin\frac{k\pi}{n}$$

Question 16 (continued)

(c) (ii) (continued)

$$|\alpha^k - 1| = |(e^{\frac{2\pi}{n}i})^k - 1|$$

$$= |e^{\frac{2k\pi}{n}i} - 1|$$

$$= |e^{\frac{k\pi}{n}i}(e^{\frac{k\pi}{n}i} - e^{-\frac{k\pi}{n}i})|$$

$$= |e^{\frac{k\pi}{n}i}(2i\,Im(e^{\frac{k\pi}{n}i}))|$$

$$= |e^{\frac{k\pi}{n}i}(2i\sin\frac{k\pi}{n})|$$

$$= |e^{\frac{k\pi}{n}i}||2i\sin\frac{k\pi}{n}|$$

$$= 1 \cdot 2|\sin\frac{k\pi}{n}| \quad *$$

$$= 2\sin\frac{k\pi}{n}$$

c) ii) COMMENT:

If we think about what we are working towards, knowing that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

We are looking to show that $|\alpha^k - 1| = 2\sin\frac{k\pi}{n}$

It is much easier to find $|z_1 z_2|$ than $|z_1 + z_2|$

since $|z_1 z_2| = |z_1||z_2|$

End of solutions